

TOPOLOGICAL INVARIANT MEANS ON $B_p^*(G)$ AND WEAKLY COMPACT MULTIPLIERS OF $A_p(G)$

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ABSTRACT. In this paper we investigate the compact and weakly compact multipliers of the Herz-algebras $A_p(G)$. Let $B_p(G)$ be the space of pointwise multipliers of $A_p(G)$. We show that there is a topological invariant mean on $B_p^*(G)$. Furthermore, we show that if $B_p^*(G)$ is separable, then G is a discrete group.

1. INTRODUCTION

Let G be a locally compact group. For $1 < p < \infty$, let $A_p(G)$ denote the Herz algebra of G and $B_p(G)$ denote the space of multipliers of $A_p(G)$. The results obtained in this paper improve some results obtained by E. E. Granirer [6] for $A_p^*(G)$. Also, we give a simple new proof for Proposition 7 of [6].

In section 3 we both define and show the existence of topological invariant means on $B_p^*(G)$. Also we show that the invariant mean is necessarily unique on $wap(B_p(G))$, the space of weakly almost periodic functional on $B_p(G)$. Furthermore, if $B_p^*(G)$ is separable then G is discrete.

In section 4 we show that if $A_p(G)$ has a [weakly] compact multiplier, then G is discrete. Furthermore, if G is amenable, then each [weakly] compact multiplier is in $A_p(G)$.

2. PRELIMINARIES AND SOME NOTATIONS

Let G be a locally compact group equipped with a fixed left Haar measure $\lambda = dx$. The spaces $L^p(G)$ ($1 \leq p \leq \infty$) have their usual meaning. Let $C_{00}(G)$, denote the space of complex continuous functions on G with compact support. Also, let $C_0(G)$ be the space of complex continuous functions on G which tend to 0 at infinity with $\|\cdot\|_\infty$ -norm. Then $M(G) = C_0(G)^*$ denotes the bounded Borel complex measures on G with convolution as multiplication, and with variation norm. For f in $L^1(G)$, the operator $\rho(f) : L^p(G) \rightarrow L^p(G)$ is defined by $\rho(f)(g) = f * g$. Clearly $\rho(f)$ is a bounded linear operator on $L^p(G)$ and by $\|\rho(f)\|$ we shall always denote the operator norm of $\rho(f)$.

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For $1 < p < \infty$ let $A_p(G)$ denote the linear subspace of $C_0(G)$ consisting of all functions of the form $u(x) = \sum_{i=1}^{\infty} g_i * f_i^{\vee}(x)$ where $f_i \in L_p(G)$, $g_i \in L_q(G)$, $\frac{1}{p} + \frac{1}{q} = 1$, $\sum_{i=1}^{\infty} \|f_i\|_p \|g_i\|_q < \infty$, $f^{\vee}(x) = f(x^{-1})$ for $x \in G$. $A_p(G)$ is a commutative Banach algebra with respect to pointwise multiplication and the norm,

$$\|u\|_{A_p(G)} = \inf \left\{ \sum_{i=1}^{\infty} \|f_i\|_p \|g_i\|_q : u = \sum_{i=1}^{\infty} g_i * f_i^{\vee} \right\}.$$

In the case $p = 2$, $A_2(G) = A(G)$ is the Fourier algebra of G , which was introduced for non commutative groups by Eymard [3]. For $p \neq 2$, $A_p(G)$ was first studied by Herz in [7].

We denote by $B_p(G)$ the set of bounded complex continuous functions u on G such that $uv \in A_p(G)$, for all $v \in A_p(G)$. The norm in $B_p(G)$ is given by

$$\|u\| = \sup \{ \|uv\|_{A_p(G)} : \|v\|_{A_p(G)} \leq 1 \}.$$

Then $B_p(G)$ is the space of pointwise multipliers of $A_p(G)$. Furthermore $B_p(G)$ is a commutative Banach algebra with respect to pointwise multiplication.

Each element $f \in L^1(G)$ defines a bounded functional ϕ_f on $A_p(G)$ by

$$\langle \phi_f, u \rangle = \int_G f(x)u(x)dx, \quad u \in A_p(G).$$

The norm of ϕ_f as an element of $A_p(G)^*$ and the operator norm of $\rho(f)$ are the same. That is,

$$\|\phi_f\| = \sup_{\|u\|_{A_p(G)} \leq 1} |\langle \phi_f, u \rangle| = \|\rho(f)\| = \sup_{\|g\|_p \leq 1} \|f * g\|.$$

It follows that $L^1(G)$ can be considered as a subspace of $A_p(G)^*$ [1]. By definition, $PM_p(G)$ denote the closure of $L^1(G)$, considered as an algebra of convolution operators on $L^p(G)$, with respect to the weak operator topology in $B(L^p(G))$, the bounded operator on $L^p(G)$. The space $PM_p(G)$ can be identified with the dual of $A_p(G)$ for each $1 < p < \infty$ [1]. If G is Abelian with dual \hat{G} , then $A_2(G) = A(G) = L^1(\hat{G})$, $B_2(G) = B(G) = M(\hat{G})$ and $PM_p(G) = L^\infty(\hat{G})$. We define the module action of $B_p(G)$ on $B_p^*(G)$ by $\langle \phi u, v \rangle = \langle \phi, uv \rangle$ for $\phi \in B_p^*(G)$, $u, v \in B_p(G)$. For $x \in G$ we show by $\chi_x = 1_{\{x\}}$ the characteristic function of $\{x\}$.

3. INVARIANT MEANS ON $B_p^*(G)$

Let $I : B_p(G) \rightarrow C, I(u) = u(e)$. Denote

$$MB_p^*(G) = \{F \in B_p^*(G) : \|F\| = F(I) = 1\}$$

and for each $x \in G$,

$$S_A^p(x) = \{u \in A_p(G) : u(x) = \|u\| = 1\}$$

and

$$S_B^p(x) = \{u \in B_p(G) : u(x) = \|u\| = 1\}.$$

Then $S_A^p(x)$ and $S_B^p(x)$ are commutative semigroup and $S_A^p(x) \subseteq S_B^p(x)$. For $x = e$, denote $S_A^p = S_A^p(e)$ and $S_B^p = S_B^p(e)$. Also if $\pi : B_p(G) \longrightarrow B_p^{**}(G)$ is the canonical map it is clear that $\pi(S_B^p) \subseteq MB_p^*(G)$.

The proof of the next theorem is similar to the proof of [6, Proposition 4] and by this theorem we can define the set of topological invariant means on $B_p^*(G)$.

Theorem 3.1. *There is $F \in MB_p^*(G)$ such that, $uF = u(e)F$, for $u \in B_p(G)$.*

Definition 3.2.

$$TIMB_p^*(G) := \{F \in MB_p^*(G) : uF = u(e)F, \text{ for all } u \in B_p(G)\}$$

is called the set of topological invariant means on $B_p^*(G)$.

Proposition 3.3. *$TIMB_p^*(G) \cap B_p(G) \neq \emptyset$ if and only if G is discrete. In this case $TIMB_p^*(G) \cap B_p(G) = \{\chi_e\}$.*

Proof. Let G be discrete. Then it is clear that $\chi_e \in TIMB_p^*(G) \cap B_p(G)$. Conversely, if $u \in TIMB_p^*(G) \cap B_p(G)$. Then $u(e) = 1$, and for each $v \in S_B^p$, $uv = u$. Let $x \neq e$. Then there is $v \in S_B^p(G)$, $v(x) = 0$. So, $u(x) = uv(x) = 0$. Hence $u = \chi_e$ and G is discrete. This show that $TIMB_p^*(G) \cap B_p(G) = \{\chi_e\}$.

The next theorem is the main result of this section. When G is a discrete commutative group, Forrest show that if $PM_p(G)$ is separable, then G is finite (see [4] Theorem 1).

Theorem 3.4. *Let G be a locally compact group and $B_p^*(G)$ be separable. Then G is discrete.*

Proof. Let u be in S_A^p with compact support. Let $E = \text{supp}u$ and $K = uS_A^p$. Then by the Markov- Kakutani fixed point theorem (see [2], p. 456) there is $F \in \{w^*clK\} \cap TIMB_p^*(G)$. Since w^* -topology on the unit ball of $B_p^{**}(G)$ is metrizable. Hence there exists a sequence $u_n \in S_A^p$ such that for all $\phi \in B_p^*(G)$, $\phi(uu_n) \longrightarrow F(\phi)$.

Now let $k \in A_p(G)$ with compact support such that $k = 1$ on E and $\psi \in PM_p(G)$. Then $k\psi$ is a bounded linear functional on $B_p(G)$ and $\psi(uu_n) = k\psi(uu_n) \longrightarrow F(k\psi)$. Therefore (uu_n) is a weakly Cauchy sequence in

$$A_E^p(G) = \{v \in A_p(G) : \text{supp}v \subseteq E\}.$$

But by Lemma 18 on p. 131 of [5] it follows that $A_E^p(G)$ is weakly sequentially complete for any compact set $E \subseteq G$. So there is $v_0 \in A_E^p(G)$ such that for each $\psi \in PM_p(G)$, $\psi(uu_n) \longrightarrow \psi(v_0)$. Hence $\psi(v_0) = F(k\psi)$. Thus for each $w \in S_A^p(G)$ we have $\psi(v_0w) = F(k\psi w) = F(k\psi) = \psi(v_0)$. So, $v_0w = v_0$. Let $x \neq e$ and $w \in S_A^p(G)$ with $w(x) = 0$. We get $v_0(x) = 0$.

However $v_0(e) = I(v_0) = F(kI) = F(I) = 1$. So $v_0 = \chi_e$. Since $v_0 \in A_p(G)$ is continuous, so G is discrete.

Theorem 3.5. *Let G be a locally compact group. Let J be a closed ideal in $B_p(G)$ such that there is $u \in J$ with $u(e) \neq 0$. Then $TIMB_p^*(G) \subseteq J^{\perp\perp}$.*

Proof Let $F \in TIMB_p^*(G)$ and $T \in J^\perp$. Since $uT = 0$. It follows that $u(e)F(T) = F(uT) = 0$. Hence $F(T) = 0$. Therefore $F \in J^{\perp\perp}$.

Let $\mu \in M(G)$ and

$$\langle u, \mu \rangle = \int_G u(x) d\mu(x) \quad \text{for } u \in B_p(G)$$

then

$$|\langle u, \mu \rangle| \leq \|\mu\| \|u\|_\infty \leq \|\mu\| \|u\|_{B_p(G)}.$$

Hence μ is a bounded linear functional on $B_p(G)$. Let $wap(B_p(G))$ denote the linear space of all weakly almost periodic functional on $B_p(G)$.

Similar to the next proposition we can give a new proof for the Proposition 7 of [6].

Proposition 3.6. $M(G) \subseteq wap(B_p(G))$.

Proof. Since $C(G)$, the linear space of bounded continuous function on G is a C^* -algebra and the unit ball of $B_p(G)$ is contained in the unit ball of $C(G)$. Now let $\mu \in M(G)$ and (f_n) and (g_m) be two sequence of the unit ball of $B_p(G)$ such that

$$\lim_n \lim_m \langle \mu, f_n g_m \rangle, \quad \lim_m \lim_n \langle \mu, f_n g_m \rangle$$

both exist. Then by Arens regularity of $C(G)$, they are equal. So, μ is a weakly almost periodic functional on $B_p(G)$.

The proof of the next proposition is similar to the Proposition 9 [6].

Proposition 3.7. *There is a unique topological invariant mean on $wap(B_p(G))$. In fact there is a unique $F \in wap(B_p(G))^*$ such that $F(I) = 1$ and $F(uT) = u(e)F(T)$, for all $u \in B_p(G)$ and $T \in wap(B_p(G))$.*

4. COMPACT AND WEAKLY COMPACT MULTIPLIERS ON $A_p(G)$

In this section we shall present some results about compact and weakly compact multiplier of $A_p(G)$. For $u \in B_p(G)$, let $\Gamma_u : A_p(G) \rightarrow A_p(G)$, defined by $\Gamma_u(v) = uv$ for $v \in A_p(G)$.

Lemma 4.1. *If for some $u \neq 0$ in $B_p(G)$, Γ_u is a weakly [compact] operator, then G is discrete.*

Proof. Let $u(x) \neq 0$ for some $x \in G$ and $K = w - cl\{uv : v \in S_A^p(x)\}$. Then K is a weakly compact convex subset of $A_p(G)$ and For each $w \in S_A^p(x)$, $\Gamma_w(K) \subseteq K$. The operators Γ_w are pairwise commuting and $(A_p(G), \text{weak})$ to $(A_p(G), \text{weak})$ continuous. By the Markov- Kakutani fixed point

theorem ([2], p.456) there is $u_0 \in K$ such that for each $w \in S^p A(x)$, $wu_0 = u_0$. Let $y \neq x$. Then there is $v \in S_A^p(x)$ with $v(y) = 0$. Hence $u_0(y) = v(y)u_0(y) = 0$. Therefore $u_0 = u(x)\chi_x$ and G is discrete.

Now, we define the set of weakly compact multipliers on $A_p(G)$ as

$$WCB_p(G) := \{u \in B_p(G) : \Gamma_u \text{ is weakly compact operator on } A_p(G)\}.$$

Theorem 4.2. *Let G be a discrete group. Then*

- (i) $A_p(G) \subseteq WCB_p(G)$.
- (ii) *Let G be an amenable group. Then $WCB_p(G) = A_p(G)$.*

Proof. (i) If $x \in G$ and $\phi = \chi_x$, then $\Gamma_\phi(u) = u(x)\phi$. Hence Γ_ϕ is compact. Since the linear span of $\{\chi_x : x \in G\}$ is dense in $A_p(G)$. It follows that $A_p(G) \subseteq WCB_p(G)$.

(ii) Let $u \in WCB_p(G)$ and (e_α) be a bounded approximate identity (B.A.I) for $A_p(G)$. Then there exist a subnet $(e_{\alpha\beta})$ and $v \in A_p(G)$ such that $e_{\alpha\beta}u \rightarrow v$ in the weak topology of $A_p(G)$. Now, let $x \in G$ and $w \in A_p(G)$ such that $w(x) = 1$. Since $e_{\alpha\beta}(uw) \rightarrow uw$ in the norm topology of $A_p(G)$. So,

$$v(x) = \lim_{\beta} (e_{\alpha\beta}u)(x) = \lim_{\beta} (e_{\alpha\beta}uw)(x) = w(x)u(x) = u(x).$$

Hence $u = v \in A_p(G)$.

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